Notes on physics

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Chapter 1

Mathematical preliminaries

1.1 Vector calculus

1.1.1 The nabla operator

in which ∇ is the three dimensional spatial derivative

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \tag{1.1}$$

The Laplacian is the scalar operator which we get by appying the nabla operator twice:

$$\nabla^2 = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial z} \tag{1.2}$$

1.1.2 Gauss theorem

$$\iiint\limits_V \nabla \cdot \mathbf{F} \, dV \quad = \quad \oint\!\!\!\oint\limits_S \mathbf{F} \cdot \mathbf{n} \, da \tag{1.3}$$

1.1.3 Stokes theorem

$$\iint\limits_A (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, da = \oint\limits_C \mathbf{F} \cdot d\mathbf{s} \tag{1.4}$$

in which $\mathbf{F}$ is an arbitrary vector field, V is an volume bounded by the surface S, and which A an area bounded by contour C.

1.2 The Hilbert space formalism

Many books on quantum mechanics begin with a long historical detour, explaining how the theory emerged from phenomena like blackbody radiation and Bohr's atomic model. Although this approach can have its charm, it also puts a lot of emphasis on how counterintuitive the theory was to physicists at century ago. Since our main goal here is to build the right physical intuition, it seems better to dive straight in.

A good starting point is on the mathematical side. The mathematical machinery of quantum mechanics—being mostly linear algebra and functional analysis—is also very useful outside quantum mechanics, for example in data science and engineering. So, it can stand pretty much on its own, and compared to the mathematics of other branches of physics, it's relatively easy. In the following section we are going to introduce the concept of *state*, which is a central concept in quantum mechanics. But for now, you could consider state to be just a generic thing, not specially tied to quantum dynamics. For example, you might just as well consider it to be the state of an audio filter or the state of a chemical plant. All that's necessary is that the state adhere to some rules described below.

1.2.1 Ket vectors

States are represented by vectors in a Hilbert space, commonly notated as *kets* in the *bra-ket* notation invented by Paul Dirac. His idea was to split an inner product, which is often notated as a bracket $\langle u|v\rangle$, in a left part $\langle u|$ called a *bra*, and a right part $|u\rangle$ called a *ket*.

Here are some examples of what kets can look like:

$|x\rangle$, $|\mathbf{p}\rangle$, $|\psi\rangle$, $|\uparrow\rangle$, $|\clubsuit\rangle$, $|+\rangle$, $|1001101\rangle$, $|0\rangle$, $|j\rangle$, etc...

The main advantage of the bra-ket notation is notational convenience. Without them you need some other typographical convention to denote that something is a vector in an abstract space rather than just a ordinary variable, like making the symbol bolt face, and the label needs to move to a subscript, which is all inconvenient, especially hundred years ago when articles where often written on a typewriter.

Although Dirac's bra-ket notation is often elegant and expressive, it can also be confusing for beginners. The main rule to avoid confusion it to treat everything inside the bracket as *just a label*. You are not supposed to do math inside a bra or a ket. You can (but shouldn't) write expressions like $|u+v\rangle$, but this has no well defined meaning, other than "$u+v$" being just another label. If you want an expression for the sum of vectors $|u\rangle$ and $|v\rangle$, this should be written as $|u\rangle+|v\rangle$. Also $|0\rangle$ is sometimes not the zero vector of the vector space, but just some ordinary state which we would like to label with the symbol "0". To avoid confusion, we will use the notation $\mathbf{0}$ for the zero vector.

For kets, the following axioms of a vector space apply:

$|u\rangle+(|v\rangle+|w\rangle)=(|u\rangle+|v\rangle)+|w\rangle$
$|u\rangle+|v\rangle=|v\rangle+|u\rangle$
There exist an element $\mathbf{0}$ (the zero vector) such that $|v\rangle+\mathbf{0}=|v\rangle$ for every vector $|v\rangle$
For every vector $|v\rangle$ there exists a vector $-|v\rangle$ such that $|v\rangle+(-|v\rangle)=\mathbf{0}$
$\alpha(\beta|v\rangle)=(\alpha\beta)|v\rangle$
$\alpha(|u\rangle+|v\rangle)=\alpha|u\rangle+\alpha|v\rangle$
$(\alpha+\beta)|v\rangle=\alpha|v\rangle+\beta|v\rangle$

So what is this telling us? First of all, we are dealing with abstract objects, which means that they have no a priori properties (besides that they adhere to the vector space axioms) unless we define them. The vector space axioms basically say that we can freely add and subtract the objects, and we can scale them with a (complex) number. That on itself is very interesting, and a remarkable number of things follow from this. But it is also good to consider what we can not do (yet) with the objects, simply because we have not defined it. For example, we can not multiply two vectors together. And we have no way of telling whetter a vector is big or small, and we can't tell yet if two vectors are similar or different.

1.2.2 Bases

In a vector space, two vectors are called dependent if one of them is a scalar multiply of the other, and independent if this is not the case. If you have a set of N vectors they are mutually independent if the following condition holds:

$$\alpha_1|u_1\rangle + \alpha_2|u_2\rangle \cdots + \alpha_N|u_N\rangle = \mathbf{0} \iff \text{all } \alpha_n = 0 \tag{1.5}$$

i.e. the only way to make the sum equal to the zero vector is to make all the coefficients zero. If you can make the sum zero with some other combination in which not all coefficients are zero, the set is called dependent.

A vector space is called N dimensional if you can find a set of N mutually independent vectors, and that adding any extra vector to the set makes the set dependent. Such a set of vectors is called a base for the space, and any other vector can be written as a linear combination of of these base vectors. So if we have a base: $\{|e_1\rangle, |e_2\rangle, |e_3\rangle \cdots |e_N\rangle\}$, we can write an arbitrary vector $|u\rangle$ as:

$$|u\rangle = \alpha_1|e_1\rangle + \alpha_2|e_2\rangle + \alpha_3|e_3\rangle \cdots + \alpha_N|e_N\rangle \tag{1.6}$$

Usually, once we established a certain base, we often don't mention the base vectors explicitly anymore and simply write the vector in column matrix notation:

$$|u\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_N \end{pmatrix} \tag{1.7}$$

We should always remember that the set of coordinates is not the vector itself, but only repents it with regard to some base. If we change base, also these coordinates change, while the vector itself remains the same.

A vector space can also have an infinite dimension. In this case you need an infinite number of coordinates to represent a vector. For the purpose of physics, infinite dimensional vectors are as easy to work with as finite dimensional vectors. Only proofs of some theorems need a bit of advanced math, but physicists can usually just skip over that.

1.2.3 Linear operators

A *linear operator* is a function on the vector space for which certain properties hold. In this part of the book we will denote linear operators with capital letters, which is common practice in mathematics. Later we will switch to the physics notation by placing a $\sim$ on top of a symbol The following properties hold for a linear operator:

$$A(|\psi\rangle + |\varphi\rangle) = A|\psi\rangle + A|\varphi\rangle$$
$$A(\alpha|\psi\rangle) = \alpha A|\psi\rangle$$
$$(A + B)|\psi\rangle = A|\psi\rangle + B|\psi\rangle$$
$$(\alpha A)|\psi\rangle = \alpha(A|\psi\rangle)$$

1.2.4 Inner product

As we said earlier, in a plain vector space there is no concept of distance. For this, we need to define the *inner product*. An inner product is a function which takes two vectors and assigns a scalar to this pair. When we are studying inner products in detail, we use the notation $(\cdots, \cdots)_{\text{inp}}$. Later on we replace it with the more elegant bra-ket notation.

So if we have a pair of vectors $|u\rangle$ and $|v\rangle$, the inner product if a function

$$(|u\rangle, |v\rangle)_{\text{inp}} \to \mathbb{C} \tag{1.8}$$

with the following properties:

$$(|u\rangle, |v\rangle)_{\text{inp}} = (|v\rangle, |u\rangle)^*_{\text{inp}}$$
$$(|u\rangle + |v\rangle, |w\rangle)_{\text{inp}} = (|u\rangle, |w\rangle)_{\text{inp}} + (|v\rangle, |w\rangle)_{\text{inp}}$$
$$(|u\rangle, \alpha|v\rangle)_{\text{inp}} = \alpha\,(|u\rangle, |v\rangle)_{\text{inp}}$$
$$(|u\rangle, |u\rangle)_{\text{inp}} >= 0$$

Note that the $*$ here denotes the complex conjugate, which means multiplying the imaginary part of a number by -1. In the following sections we will generalize the meaning of the $*$ operator.

From the definition directly follows the following:

$$(\alpha|u\rangle, |v\rangle)_{\text{inp}} = \alpha^*\,(|u\rangle, |v\rangle)_{\text{inp}} \tag{1.9}$$

and

$$(\alpha|u\rangle, |v\rangle)_{\text{inp}} = (|u\rangle, \alpha^*|v\rangle)_{\text{inp}} \tag{1.10}$$

1.2.5 The adjoint of a linear operator

In the previous section we have seen that a scalar can switch places in the inner product if we conjugate it. This phenomena can be generalized to linear operators (scalar multiplication is a special case of a linear operator) in the following way:

For every linear operator A, there is a linear operator A^* so that

$$(A|u\rangle, |v\rangle)_{\text{inp}} = (|u\rangle, A^*|v\rangle)_{\text{inp}} \tag{1.11}$$

The operator A^* is called the adjoint of A.

From this definition and that of the inner product, it's easy to derived that $A^{**} = A$, and $(AB)^* = B^*A^*$.

1.2.6 Bra vectors

A special subset of linear operators are those which map a vector to a scalar. We call them *linear functionals*. So if F is a linear functional and $|u\rangle$ a vector, then $F|u\rangle$ is a complex number.

The set of all linear functionals is itself a vector space. So if $\mathcal{H}$ is our original vector space, there always exist a second vector space $\mathcal{H}^*$, called the *dual space*, which contains all linear functionals which can act on vectors from $\mathcal{H}$. For finite dimensional vector spaces the dimensions of $\mathcal{H}$ and $\mathcal{H}^*$ are the same. This means that we can make a one-to-one mapping between all elements of both spaces. It turns out that for infinite dimensional vector space a similar thing can be done.

The elements from the dual space are denoted with so called *bras*, which look like $\langle label|$. When we apply it to a ket, we get an expression like $\langle label||u\rangle$, in which the two adjacent vertical bars are merged into one: $\langle label|u\rangle$.

The application of bras to kets does something very similar as the inner product does, i.e. mapping pairs of vectors to scalars. Now, the idea of the bra-ket notation is that we can make these two concepts *exactly* the same by using the following convention: We give the bras the same labels as the kets (we have the same number of both of them). And we asign the labels to the bras in such way that they produce the same number as the inner product does, so

$$\langle u|v\rangle = (|u\rangle, |v\rangle)_{\text{inp}} \tag{1.12}$$

From now on, the bra-ket *is* the inner product, and bras and kets have become essentially the same thing.

An alternative way to write$\langle u|$ is $|u\rangle^*$, so

$$|u\rangle^* = \langle u| \tag{1.13}$$

1.2.7 Length and angles

The definition of the inner product is obviously inspired by the geometrical properties of ordinary three dimensional space. Therefore the concepts of length and angle can readily be generalized to abstract spaces:

We define the *length* of a vector as the square root of the inner product with itself, i.e:

$$||u\rangle| = \sqrt{\langle u|u\rangle} \tag{1.14}$$

We can also define the *angle* between two vectors. Say, we have vectors $|u\rangle$ and $|v\rangle$. If we calculate the squared lenght of the difference between these vectors we get

$$||u\rangle - |v\rangle|^2 = (|u\rangle - |v\rangle, |u\rangle - |v\rangle)_{\text{inp}} = ||u\rangle|^2 + ||v\rangle|^2 - 2\langle u|v\rangle \tag{1.15}$$

Now, if these vectors would have been ordinary line segments in three dimensional space, the cosine rule would give:

$$||u\rangle - |v\rangle|^2 = ||u\rangle|^2 + ||v\rangle|^2 - 2||u\rangle|\ ||v\rangle| \cos(\theta) \tag{1.16}$$

If we combine those expression we get:

$$\langle u|v\rangle = ||u\rangle|\ ||v\rangle| \cos(\theta) \tag{1.17}$$

So we could generalize the concept of angle as:

$$\cos(\theta) = \frac{\langle u|v\rangle}{||u\rangle|\ ||v\rangle|} \tag{1.18}$$

1.2.8 Orthogonal bases

If we select base vectors, it is almost always desirable to chose vectors which are orthogonal and have a length of 1. So if we have a base $\{|e_1\rangle, |e_N\rangle, \cdots |e_N\rangle\}$, we require:

$$\langle e_m|e_n\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{1.19}$$

In any base, orthogonal or not, we can write the inner product in terms of the coordinates of the vectors:

$$\langle v|u\rangle = (v_1\langle e_1| + v_2\langle e_2| + \cdots v_N\langle e_N|)\ (u_1|e_1\rangle + u_2|e_2\rangle + \cdots u_N|e_N\rangle) = \sum_m \sum_n v_m u_n \langle e_m|e_n\rangle \tag{1.20}$$

But if the base is orthogonal, this sum simplifies to:

$$\langle v|u\rangle = \sum_n v_n u_n = (v_1, v_2, \cdots v_N) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \tag{1.21}$$

in which we recognize the well known matrix multiplication.

1.2.9 Decomposition in base vectors

Say we have a discrete set of base vectors: $|1\rangle, |2\rangle, |3\rangle, \cdots$ which span the space. We also demand that these vectors are orthogonal with respect to each other. This means that the inner product between any pair $|n\rangle$ and $|m\rangle$ of these base vectors is zero, and their length is one:

$$\langle n|m\rangle = \delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \tag{1.22}$$

in which δ_{nm} is called the Kronecker-delta.

Often the set of base vectors is a continuum. Say, we have an infinite set of base states $|x\rangle$ in which x can be any real number. Orthogonality between any pair $|x\rangle$ and $|y\rangle$ of base vectors in a continuous basis means:

$$\langle x|y\rangle = \delta(x - y) \tag{1.23}$$

in which $\delta()$ is the Dirac delta function.

In the remainder of this section, let $|n\rangle$ denote a discrete base vector, $|x\rangle$ a continuous base vector, and $|u\rangle$ an arbitrary state, i.e. an arbitrary sum of scaled base vectors. With the inner product we can measure how much $|u\rangle$ overlaps with any of the base vectors. This way, we can form a function:

$$u[n] = \langle n|u\rangle \tag{1.24}$$

or, in the continuous case:

$$u(x) = \langle x|u\rangle \tag{1.25}$$

The functions $u[n]$ and $u(x)$ are called *wave functions*. They represent the state with respect to some base. It is important to note that these wave functions are not unique for a given state, because we are free to choose another set of base vectors, and whenever we do, the wave function changes. Also note how the symbol u now has two subtle different meaning; in the ket $|u\rangle$ it is the name of the quantum state, and in the function $u[n]$ and $u(x)$ its the name of a wave function which *represents* the quantum state in a certain basis. Ideally a different symbol should have been used, but because of the shortage of symbols in general it is usually more economical to reuse the same symbol for both the state and the wave function. But it can become even more confusing when we consider multiple basis at the same time. Then we can have multiple different wave functions which all represent the same state. For example we can express the state $|u\rangle$ as a function of position $u(x)$, or as a function of momentum $u(p)$ and some authors use the same symbol u to denote both these functions, while they are actually completely different function. They expect the reader to infer from the parameter (x or p) to understand which of the two is meant.

Now that we know how to measure how much each base vectors contribute to a certain state, we can decompose a state as the sum (or integral) of the base vectors:

$$|u\rangle = \sum_n u[n]|n\rangle \tag{1.26}$$

or in continuous form:

$$|u\rangle = \int u(x)|x\rangle dx \tag{1.27}$$

1.2.10 Operators applied to bras

The multiplications between operators and bras and kets behaves associative:

$$\langle u| \left(A|v\rangle\right) = \left(\langle u|A\right)|v\rangle = \langle u|A|v\rangle \tag{1.28}$$

Note that with the bra-ket notation for the inner product, we don't require the explicit application of the $*$ to A, because the conjugation is implied by the fact that A works *from the right* on $\langle u|$.

We also have:

$$(A|u\rangle)^* = \langle u|A^* \tag{1.29}$$

and

$$((\langle u|A)^* = A^*|u\rangle \tag{1.30}$$

1.2.11 Unitary and Hermitian operators

In section 1.2.5 we have defined the adjoint of an operator. In this section we are going to define two important classes of operators with very specific behavior with respect to adjoint.

The first one are the operators for which the adjoint is equal to its inverse. Such operators are called unitary:

$$U^* = U^{-1} \quad \Leftrightarrow \quad U \text{ is unitary} \tag{1.31}$$

Unitary operators are very important in quantum mechanics, because they describe how the quantum system goes from one state to another. The first property, which directly follows from its definition, is that unitary operators are always invertible. This also means quantum states evolve in a way which is in principle always invertible (no information ever gets lost). Their second property is that unitary operators preserve inner products, so:

$$(U\varphi, U\psi)_{\text{inp}} = \langle\varphi|U^*U|\psi\rangle = \langle\varphi|\psi\rangle \tag{1.32}$$

The geometrical interpretation of this is that unitary transforms are rotations and reflections.

The second group of operators which deserve special attention are the so called *Hermitian* operators. They have the property that they are equal to their own adjoint:

$$A^* = A \tag{1.33}$$

Because of this defining property these operators are also often called *self-adjoint*. In the inner product, the Hermitian operator is allowed to switch places:

$$(A\varphi, |\psi\rangle)_{\text{inp}} = (\varphi, A|\psi\rangle)_{\text{inp}} \tag{1.34}$$

In matrix form Hermitian operators are easy to recognize; they are symmetrical in the main diagonal (and complex matrix coefficients being conjugated).

Hermitian and unitary operators are related in a very elegant way. If A is Hermitian, then it's exponential is unitary:

$$U = \exp(jA) \tag{1.35}$$

You can easily see that U is unitary by multiplying both sides of the equation with its complex conjugate, and using the fact that $\exp(A)^* = \exp(A^*)$, and that the conjugate of j is $-j$.

Besides linking the concepts of unitarity and self-adjontness, this exponential form is also interesting because it's the solution of a first order linear differential equation

$$\frac{d}{dt}|\psi(t)\rangle = jA|\psi(t)\rangle \tag{1.36}$$

which has solution

$$|\psi(t)\rangle = \exp(jAt)|\psi(0)\rangle \tag{1.37}$$

This solutions says that the state $|\psi\rangle$ evolved via an unitary transform from some initial state.

1.2.12 Projectors

We have seen that we can construct an Hermitian operator by multiplying an operator with its own conjugate. A bra is also an operator, as it sends kets to scalars in a linear way, so we could construct an Hermitian operator from a bra.

Let's do that with $\langle e|$. If we multiply it from the right with its conjugate, we get the inner product $\langle e|e\rangle$, which is a scalar, and as such the simplest possible (Hermitian) operator. But a much more interesting form arises if we multiply $\langle e|$ from the *left* side whit it conjugate. We then get:

$$P = \langle e|^* \langle e| = |e\rangle\langle e| \tag{1.38}$$

It's easy to check that P is Hermitian. If we furthermore demand that $|e\rangle$ is normalized (i.e. $\langle e|e\rangle = 1$, then P becomes a so called projector.

A projector is *idempotent*, which means that after you have applied it once, it keeps giving the same result if you apply it again. We can write this property as

$$P^2 = |e\rangle\langle e||e\rangle\langle e| = P \tag{1.39}$$

Chapter 2

Basic quantities

2.1 Mechanics

2.1.1 Momentum

Momentum is a measure for the amount of movement. For non-relativistic speeds, its the product of mass and velocity,

$$\mathbf{p} = m\mathbf{v} \tag{2.1}$$

The relativistic expression is:

$$\mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1 - |\mathbf{v}|^2/c^2}} \tag{2.2}$$

The time derivative of the momentum is the force acting on the object:

$$\mathbf{F} = \frac{d}{dt}\mathbf{p} \tag{2.3}$$

2.1.2 Angular Momentum

Besides the linear form, momentum also has a rotational form, which is *defined* as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{2.4}$$

There is also an rotational analog of *force* which we call *torque*, which is defined as the time derivative of the angular momentum

$$\boldsymbol{\tau} = \frac{d}{dt}\mathbf{L} \tag{2.5}$$

Since the time derivative is distributive over the cross product, it readably follows that

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{2.6}$$

2.1.3 Rotational inertia

Let say we have a mass which is connected to a beam that can rotate on an axis. If we push the beam, it will start to rotate, and if we stop pushing it will keep rotating if there is no friction. This circular motion of the mass is also explained by Newtons laws. The first law states that the mass will keep moving in a straight line if there is no force applied to it. But in this case the beam is applying a force to the mass which is directed at the center of rotation. The second law states that this force will make the mass accelerate towards the center of rotation. And this is exactly how the mass is moving: It has a velocity v tangentially to the beam because of the first law, and an acceleration towards the center because of the second law.

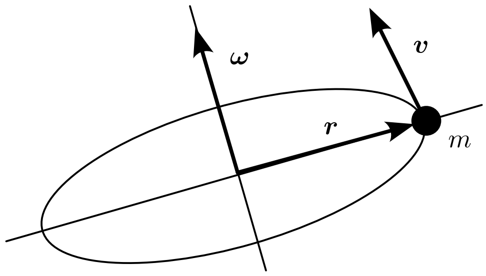

Figure 2.1: Rotational inertia.

2.2 Planck's constant

It was first proposed by Max Planck that light is emitted in packets of a fixed amount of energy, and that the amount of energy of each packet is equal to the frequency of the light. If we want to express the energy in Joules, a conversion factor h is needed, which is called Planck's constant

$$E = nhf \quad \text{with } n \in \mathbb{N} \tag{2.7}$$

in which $h = 6.62607015 \times 10^{-34}\ J/Hz$.

For $n = 1$ we have the energy of a single photon.

Planck's constant could be seen as the bridge between everyday-object scale and quantum scale. The fact that it's very small explains why we normally not experience quantum effects. Originally it's value was measured in experiments, but since the redefinition of SI constants in 2019 it is defined as exactly this number.

In equations, Planck's constant is often found next to 2π, so to make notation more compact, physicist introduced the so called reduced Planck constant (h-bar)

$$\hbar = \frac{h}{2\pi} \tag{2.8}$$

2.3 de Broglie wavelength

It was first proposed in 1923 by Louis de Broglie that not only photons, but *all* particles have a wave length, which is given by

$$\lambda = \frac{h}{|\mathbf{p}|} \tag{2.9}$$

in which λ is the wavelength in meter, $\mathbf{p} = m\mathbf{v}$ the momentum of the particle with mass m and velocity $\mathbf{v}$.

By expressing the wave length as a scalar, we loose useful information about the direction of the wave. It is therefore often handier to work with a closely related quantity, the wave number $\mathbf{k}$, which is a vector with points in the direction of the propagation of the wave thought space, and a magnitude inversely proportional to the wave length,

$$\mathbf{k} = \frac{\mathbf{p}}{\hbar} \tag{2.10}$$

Note that in a wave function, which in general looks something like

$$u(\mathbf{r}, t) = \sin(\omega t - \mathbf{k} \cdot \mathbf{r}) \tag{2.11}$$

the wave number $\mathbf{k}$ has an analog role in the space dimension as the angular frequency ω has in the time dimension, i.e. ω is the number of waves per 2π seconds, and $|\mathbf{k}|$ the number of waves per 2π meter.

2.3.1 Momentum of a photon

The energy of a single photon is

$$E = hf = \hbar\omega = c|\mathbf{p}| \tag{2.12}$$

The momentum of a photon is:

$$\mathbf{p} = \hbar\mathbf{k} \tag{2.13}$$

2.4 Energy

2.4.1 Electron volt

1 eV is the amount of energy which is required to move an electron over a 1V potential difference.

1.602176634 E−19 J

2.4.2 Relativistic energy of a particle

$$E = \gamma m_0 c^2 \tag{2.14}$$

in which m_0 is the rest mass of the particle, and

$$\gamma = \frac{1}{\sqrt{1 - |\mathbf{v}|^2/c^2}} \tag{2.15}$$

the Lorentz factor. For a stationary particle this factor is exactly 1, and it remains close to 1 for ordinary speeds. Only for speeds comparable to the speed of light the factor begins to grow, going to infinity as the particle approaches the speed of light.

$$E_{\text{rest}} = m_0 c^2 \tag{2.16}$$

A different equation for the total energy is this one:

$$E^2 = (m_0 c^2)^2 + (pc)^2 \tag{2.17}$$

For a particle which close to the speed of light we have:

$$E \approx |\mathbf{p}|c \tag{2.18}$$

And for particles at normal speeds we have:

$$E \approx m_0 c^2 + \frac{|\mathbf{p}|^2}{2m_0} \tag{2.19}$$

Chapter 3

Electrodynamics

3.1 The Maxwell equations in free space

3.1.1 Physical quantities

F= Force in N (Newton)

v= velocity in m/s

q = Electric charge in C (Coulomb)

E= Electric field strength in V/m or N/C

B= Magnetic flux density in T (tesla) or Wb/m^2 or [kg $\mathrm{s}^{-2}\mathrm{A}^{-1}$]

j= Current density in A/m^2

ρ = Charge density in C/m^3

c = Speed of light in free space, which is defined to be 299792458 m/s

μ_0 = permeability of free space, which is defined to be $4\pi \cdot 10^{-7}\ H/m$ or N/A^2

ε_0 = permittivity of free space, which is defined as $\frac{1}{c^2\mu_0} \approx 8.85419 \cdot 10^{-12}\ F/m$

3.1.2 Densities

Currents and charges can be found by integrating their densities:

$$I_{throug\ A} = \iint\limits_A \mathbf{j} \cdot \mathbf{n}\, da \qquad Q_{inside\ V} = \iiint\limits_V \rho\, dV \tag{3.1}$$

3.1.3 Lorentz force

$$\mathbf{F} = q\,(\,\mathbf{E}\; + \;\mathbf{v} \times \mathbf{B}\,) \tag{3.2}$$

3.1.4 The Maxwell equations in differential form

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho \qquad \text{(Gauss' law)} \tag{3.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \qquad \text{(Faraday's law)} \tag{3.4}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No magnetic monopoles)} \tag{3.5}$$

$$c^2 \nabla \times \mathbf{B} = \frac{1}{\varepsilon_0}\mathbf{j} + \frac{\partial}{\partial t}\mathbf{E} \qquad \text{(Ampere's law with Maxwell's correction)} \tag{3.6}$$

From these equations immediately follows the conservation of charge law:

$$\nabla \cdot \mathbf{j} \;\; = \;\; -\frac{\partial}{\partial t}\rho \tag{3.7}$$

3.1.5 The Maxwell equations in integral form

With the Gauss and Stokes equations, the differential form Maxwell equations can be re-written in integral form

$$\oiint\limits_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\varepsilon_0} Q \tag{3.8}$$

$$\oint_C \mathbf{E} \cdot \, d\mathbf{s} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n} \, da \tag{3.9}$$

$$\oiint_S \mathbf{B} \cdot \mathbf{n} \, da = 0 \tag{3.10}$$

$$c^2 \oint_C \mathbf{B} \cdot \, d\mathbf{s} \quad = \quad \frac{1}{\varepsilon_0} I \, + \, \frac{\partial}{\partial t} \iint_S \mathbf{E} \cdot \mathbf{n} \, da \tag{3.11}$$

3.2 Polarization and Magnetization

When working with the Maxwell equations in a real world situations, it can be convenient to divide the charge and currents into a macroscopic part that we can directly control, and a microscopic part that happens inside matter beyond our direct control.

For the charge density we have

$$\rho_{tot} \quad = \quad \rho_{free} \quad + \quad \rho_{pol} \quad = \quad \varepsilon_0 \, \nabla \cdot \mathbf{E} \tag{3.12}$$

In which ρ_{free} is the charge density that we apply by moving (displacing) free charges. This will give a field that is called the electric displacement field **D**. In free space the **D** field is just the **E** field expressed in a different unit. If the **D** field goes through matter, the bounded charges inside the atoms and molecules will slightly shift in the direction of **D**. This will cause a polarization field **P** that is roughly in the opposite direction as **D**. So, the total field strength **E** becomes somewhat less than it would be in vacuum.

So we have

$$\rho_{free} = \nabla \cdot \mathbf{D} \quad and \quad \rho_{pol} = -\nabla \cdot \mathbf{P} \tag{3.13}$$

and

$$\mathbf{E} = \frac{1}{\varepsilon_0}(\mathbf{D} - \mathbf{P}) \tag{3.14}$$

For currents we have

$$\mathbf{j}_{tot} \quad = \mathbf{j}_{free} \, + \, \mathbf{j}_{mag} \, + \, \mathbf{j}_{pol} \quad = \quad \frac{1}{\mu_0} \nabla \times \mathbf{B} \, - \, \varepsilon_0 \frac{\partial}{\partial t} \mathbf{E} \tag{3.15}$$

in which $\mathbf{j}_{free}$ is the current over which we have control, and which typically runs through wires. This external current gives a field that is called the magnetic field strength **H**. In free space the **H** field is just the **B** field expressed in a different unit. If the **H** field goes through matter it can cause small currents inside atoms denoted by $\mathbf{j}_{mag}$ which will produce a magnetization field **M**. Besides the magnetization there can be small currents called $\mathbf{j}_{pol}$ which are caused by a changing polarization.

So we have

$$\mathbf{j}_{free} = \nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D}, \quad \mathbf{j}_{mag} = \nabla \times \mathbf{M} \quad \textit{and} \quad \mathbf{j}_{pol} = \frac{\partial}{\partial t}\mathbf{P} \tag{3.16}$$

and

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{3.17}$$

3.2.1 The Maxwell equations inside matter

Physical quantities

D= Electric Displacement field or electric flux density in C/m^2

H= Magnetic field strength or auxiliary field in A/m

P= Polarization density in C/m^2

M= Magnetization in A/m

S= Poynting vector in W/m^2

Poynting vector (Energy flux)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{3.18}$$

The Maxwell equations in differential form

$$\nabla \cdot \mathbf{D} = \rho_{free} \tag{3.19}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{3.20}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.21}$$

$$\nabla \times \mathbf{H} = \mathbf{j}_{free} + \frac{\partial}{\partial t}\mathbf{D} \tag{3.22}$$

The Maxwell equations in integral form

$$\oiint_S \mathbf{D} \cdot \mathbf{n}\, da = Q_{free} \tag{3.23}$$

$$\oint_C \mathbf{E} \cdot\, d\mathbf{s} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n}\, da \tag{3.24}$$

$$\oiint_S \mathbf{B} \cdot \mathbf{n}\, da = 0 \tag{3.25}$$

$$\oint_C \mathbf{H} \cdot\, d\mathbf{s} \quad = \quad I_{free} \; + \; \frac{\partial}{\partial t} \iint_S \mathbf{D} \cdot \mathbf{n}\, da \tag{3.26}$$

3.3 Coulomb's law

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|} \tag{3.27}$$

Note that $4\pi|\mathbf{r}|^2$ is the surface area of a sphere with radius $\mathbf{r}$, so apart from the conversion factor ϵ_0, the magnitude of the force could be seen as the product of the two charges, deluded over the area of a sphere at distance $\mathbf{r}$. The factor $\mathbf{r}/|\mathbf{r}|$ is just a unit vector which defines the direction of the force.

Chapter 4

Quantum Mechanics

In chapter 2 we introduced the concept of the Hilbert space as the mathematical frame work of Quantum mechanics, and we emphasized that in that chapter the vectors and states should be regarded as generic abstract objects. In this chapter, we are going to use the Hilbert space frame work to study quantum mechanics, so now it would be an appropriate time to reveal what the vectors actual mean. Unfortunately, the answer is quite disappointing and even a bit disturbing: no one knows.

We do know, however, what the meaning is of the squared magnitude of the inner product of two states: its a *probability*:

$$|\langle a|b\rangle|^2 = \text{Prob}\{\text{We measure state } |a\rangle \text{ when the system before measurement was in state } |b\rangle\} \tag{4.1}$$

4.1 Measurements

The reason that the true meaning of the quantum state remains a mystery, is that nature seems to take special care to never reveal it to us. This becomes clear if we take a close look at quantum measurements. In classical physics, we can design the measurement apparatus in such way that it hardly disturbs the object we are measuring. In quantum measurements this is not possible; every measurement of the system has a profound influence on the system we measure. This influence is not caused by careless design of the measurement, but it's a law of nature for which there is no work-around.

In quantum dynamics a measurement always works by projecting the state onto some orthogonal basis. Say we have measurement basis $\{|a_1\rangle, |a_2\rangle, \cdots |a_N\rangle\}$, and $|\psi\rangle$ is the current state of the system that we want to measure. We can write $|\psi\rangle$ as a weighted sum of the measurement base:

$$|\psi\rangle = \psi_1|a_1\rangle + \psi_2|a_2\rangle + \cdots \psi_N|a_N\rangle \tag{4.2}$$

At the moment of measurement, the state $|\psi\rangle$ *collapses* randomly to one of the base vectors. The chance that the state collapses to a base vector is equal to the squared magnitude of the coefficient in front of that base vector in (4.2), i.e.

$$\text{Prob}\{|\psi\rangle \to |a_n\rangle\} = |\langle a_n|\psi\rangle|^2 = |\psi_n|^2 \tag{4.3}$$

Note that since one of the outcomes must occur, the total chance of all outcome must sum to 1:

$$\sum_n |\psi_n|^2 = 1 \tag{4.4}$$

We can describe the process of measurement by applying a projector operator $P_n = |a_n\rangle\langle a_n|$ to the state:

$$P_n|\psi\rangle = \psi_n|a_n\rangle \tag{4.5}$$

To make this resulting state properly normalized, we can simple remove the coefficient ψ_n, so the state after measurement becomes just $|a_n\rangle$. Note that with respect to our current measurement base, the state is no longer a super position. So if we measure it again in the same base, we keep getting the same out come, i.e. the state remain $|a_n\rangle$.

However, if we measure in a different (incompatible) base, the randomness comes back. For example, we could express the state in a base $\{|b_1\rangle, |b_2\rangle, \cdots |b_N\rangle\}$ as:

$$|a_n\rangle = \varphi_1|b_1\rangle + \varphi_2|b_2\rangle + \cdots \varphi_N|b_N\rangle \tag{4.6}$$

and if we measure in this base, the state randomly collapses to one of the vectors $|b_m\rangle$ with probability $|\varphi_m|^2$.

There is one more thing that we need for a measurement; we need to define what the measurement apparatus is going to display as a result (because the state itself always remains an abstract invisible thing). In principle, a quantum measurement always yields a real number, that is, with every base state $|a_n\rangle$ we associate a real number a_n which is shown on the display if the state collapses to $|a_n\rangle$.

So, the measurement of a quantum state is a probabilistic process, which we can summarize as follows:

$$\begin{aligned}
&\text{Prob}\{\text{display shows } a_n \text{ after measurement}\}\\
&= \text{Prob}\{\text{nature has randomly chosen projector } P_n \text{ and applied it to } |\psi_n\rangle\}\\
&= \text{Prob}\{\text{state collapses to } |a_n\rangle\}\\
&= |\langle a_n|\psi\rangle|^2 \;=\; \langle\psi|P_n|\psi\rangle \;=\; |\psi_n|^2
\end{aligned}$$

4.1.1 Observables are Hermitian operators

The collection of measurement outcomes $\{a_1, a_2, \cdots a_N\}$ and the measurement base $\{|a_1\rangle, |a_2\rangle, \cdots |a_N\rangle\}$ together are equivalent to an Hermitian operator $\hat{a}$ which has the base vectors $|a_n\rangle$ as eigenvectors with the measurement outcomes a_n as eigenvalues. (From now on we denote operators by placing a little *hat* on top of a symbol).

An eigenvector of an operator is a vector which does not change its direction when the operator works on it, i.e. the operator only scales it with some factor:

$$\hat{a}|a_n\rangle = a_n|a_n\rangle \tag{4.7}$$

The factor by which the eigenvector is scaled is called the eigenvalue.

We can construct $\hat{a}$ from the set of outcomes and the set of measurement base vectors as follows:

$$\hat{a} = \sum_n a_n P_n \quad \text{with} \quad P_n = |a_n\rangle\langle a_n| \tag{4.8}$$

It is important to note that this operator does not actually play an active role during a measurement, i.e. you do not *apply* this operator to anything during measurement. The operator $\hat{a}$ only represents the observable quantity in a sort of formal way.

4.2 The Schrödinger equation

Depending on the exact context, the schrödinger equation can be written in slightly different forms. One of the most general forms is:

$$j\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \tag{4.9}$$

in which $\hat{H}(t)$ the Hamiltonian operator. In clasical physics, the Hamiltonian is a function which represents the total energy of a system, which is the sum of the kinetic energy and the potential energy. In quantum mechanics, this is similar, but instead of a function, the Hamiltonian is now an operator

$$\hat{H}(t) = \hat{T} + \hat{V}(t) \tag{4.10}$$

in which $\hat{T}$ is the kinetic energy operator, and $\hat{V}(t)$ the potential energy operator. Note that $\hat{T}$ is almost never time dependent.

4.3 Single particle

For a single particle with mass m the kinetic energy operator, in turn, can be written as:

$$\hat{T} = \frac{\hat{p}^2}{2m} \tag{4.11}$$

in which $\hat{p}$ is the momentum operator, an m the mass of a particle.

4.3.1 In position basis

Depending on the basis in which we work, the momentum operator can take different forms. If we work in position basis, it is the derivative with respect to position:

$$\hat{\mathbf{p}} = -j\hbar\nabla \tag{4.12}$$

The eigenfunctions of the momentum operator are plain wave functions, i.e. functions of the form

$$\psi_n(\mathbf{r}, t) = \exp(j\omega t - j\mathbf{p}\,\mathbf{r}/\hbar) \tag{4.13}$$

We can verify that this function is indeed an eigenfunction by applying the momentum operator to is:

$$\hat{\mathbf{p}}\,\psi_n(\mathbf{r}, t) \;=\; \mathbf{p}\,\psi_n(\mathbf{r}, t) \tag{4.14}$$

For the kinetic energy operator we have:

$$\hat{T} = -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) = -\frac{\hbar^2}{2m}\nabla^2 \tag{4.15}$$

in which ∇^2 is the Laplace operator.